INSTITUT TEKNOLOGI BANDUNG

Reduced Bilinear Control System using H-infinity Balancing

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Why Bilinear System?

- Problem in real world: Nonlinear model
- There is not a standard procedure to design a control system (generally it is approached by numerical method)
- Solved by linearization (some informations may be lost)
- The other approach: Bilinearization
- Bilinear systems are the simplest class of nonlinear systems
- Naturally found in electrical networks, mechanical links, surface vehicle, heat transfer, nuclear fission, airplanes, fluid flow, socioeconomics, chemistry, demography, immunology, agriculture, respiratory chemostat, cardiovascular regulator, hormone regulation, urban processes, kidney water balance, combat model, predator-prey models

Bilinear Model

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Where: $x(t) \in \mathbb{R}^n$ is the state vector

$u(t) \in \mathbb{R}^m$ is the control input

$y(t) \in \mathbb{R}^p$ is the output of system

u_i is the i -th element of $u(t)$

$$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{n\times p},\ N_i \in \mathbb{R}^{n\times n}$$

$$i = 1, 2, \ldots, m$$

Model of Cell-Cycle Specific Cancer Chemotherapy

α: transition rate from P to Q

β: transition rate from Q to P

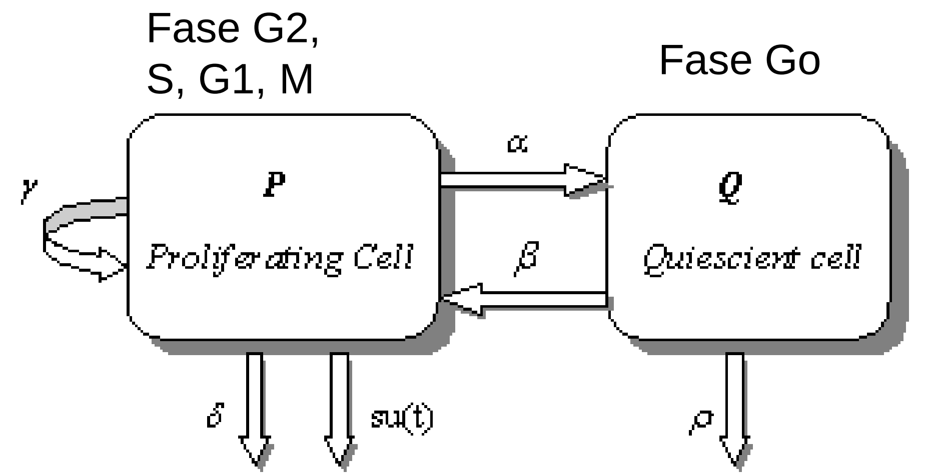
γ: cycling cells growth rate

δ: natural cell death

ρ: cell differentiation

s: effectiveness of the treatment

u: control describing the effects of the chemotherapeutic treatment only on the proliferating cells



$$\dot{P} = \left(\gamma - \delta - \alpha - su(t)\right)P + \beta Q,\ P(0) = P_0$$

$$\dot{Q} = \alpha P - \left(\rho + \beta\right)Q,\ Q(0) = Q_0$$

We design the control functional to maximize both the bone marrow mass and dose over the treatment interval.

Bilinear Optimal Control Problem

To find control $u(t)$:

$$\dot{N}(t) = \big(A + u(t)B\big)N(t),\ N(0) = N_0$$

where $A = \begin{pmatrix} \gamma - \delta - \alpha & \beta \\ \alpha & -(\rho + \beta) \end{pmatrix}$ $B = \begin{pmatrix} -s & 0 \\ 0 & 0 \end{pmatrix}$

So that maximize $J(u) = \int_0^T \left[a\big(P(t) + Q(t)\big) - \frac{b}{2}\big(1 - u(t)\big)^2 \right] dt$

By using Pontryagin maximum priciple

$$u^*(t) = \frac{b - \lambda_1 sP + w_1(t) - w_2(t)}{b}$$

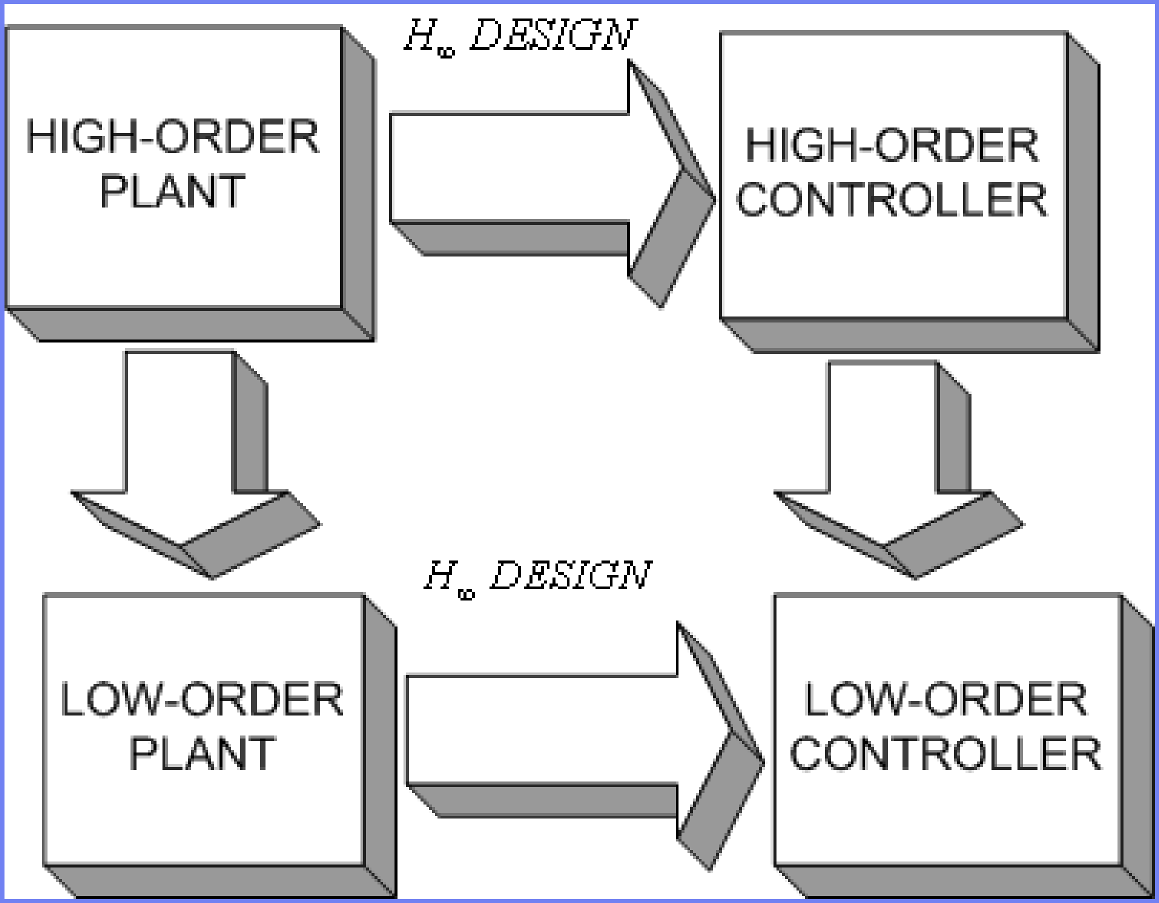
$$u^*(t) = \min\left\{1, \left(\frac{b - \lambda_1 sP}{b}\right)^+\right\}$$

$$\left(\frac{b - \lambda_1 sP}{b}\right)^+ = \begin{cases} \frac{b-\lambda_1 sP}{b},\ jika\ \frac{b-\lambda_1 sP}{b} > 0 \\ 0,\ jika\ \frac{b-\lambda_1 sP}{b} \le 0 \end{cases}$$

Model Order Reduction

- Curse of dimensionality
  - Uncertainties : we can not predict exactly what the output of a real physical system will be even if the input is known.
  - Numerical difficulties
  - High computational cost
- A low-order controller is needed for saving hardware resources and avoiding numerical difficulties in the digital processing
- Designing robust control leads to a high-order controller

BASIC APPROACH TO REDUCE THE ORDER OF CONTROLLER



Bilinear Model Reduction

Consider a nth-order bilinear system

$$G:\begin{cases}\dot{x}(t)=Ax(t)+\sum_{k=1}^{m}N_k x(t)u_k(t)+Bu(t)\\ y(t)=Cx(t)\end{cases}$$

We will find the r th-order bilinear system ($r<n$)

$$G_r:\begin{cases}\dot{x}_r(t)=A_r x_r(t)+\sum_{k=1}^{m}N_{rk} x_r(t)u_k(t)+B_r u(t)\\ y_r(t)=C_r x_r(t)\end{cases}$$

Such that $\left\|G-G_r\right\|_{\infty}$ minimum or $\left\|G-G_r\right\|_{H_2}$

Balanced Bilinear System

The controllability grammian $P = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P_i P_i^T dt_1 \cdots dt_i$

where $P_1(t_1) = e^{At_1} B, \; P_i(t_1, \ldots, t_i) = e^{At_i} \begin{bmatrix} N_1 P_{i-1} & N_2 P_{i-1} & \cdots & N_m P_{i-1} \end{bmatrix}, \; i = 1, 2, 3, \cdots$

$$AP + PA^T + \sum_{k=1}^{m} N_k P N_k^T + BB^T = 0$$

The observability grammian $Q = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} Q_i^T Q_i dt_1 \cdots dt_i$

$$Q_1(t_1) = Ce^{At_1} \quad Q_i(t_1, \cdots, t_i) = \begin{bmatrix} Q_{i-1} N_1 \\ Q_{i-1} N_2 \\ \vdots \\ Q_{i-1} N_m \end{bmatrix} e^{At_i}$$

$$A^T Q + QA + \sum_{k=1}^{m} N_k^T Q N_k + C^T C = 0$$

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} (\gamma N_k) x(t) \left(\frac{u_k(t)}{\gamma} \right) + (\gamma B) \left(\frac{u(t)}{\gamma} \right)$$

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} \tilde{N}_k x(t) \tilde{u}_k(t) + \tilde{B}\tilde{u}(t)$$

$$AP + PA^T + \gamma^2 \sum_{k=1}^{m} N_k P N_k^T + BB^T = 0$$

$$A^T Q + QA + \gamma^2 \sum_{k=1}^{m} N_k^T Q N_k + C^T C = 0$$

The balanced system

$$x_b'(t) = A_b x_b(t) + \sum_{i=1}^{m} N_{bi} x_b(t) u_i(t) + B_b u(t)$$

$$y(t) = C_b x_b(t)$$

where $A_b = T^{-1}AT, B_b = T^{-1}B, N_{bi} = T^{-1}N_i T, C_b = CT$

Controllability Grammian $P_b = T^{-1} P T^{-T}$

Observability Grammian $Q_b = T^T P T$

$$P_b = Q_b = \Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

$$\sigma_i = \left(\lambda_i (PQ)\right)^{1/2}$$

Balanced Truncation

Partition the balanced system as

$$\begin{bmatrix} x'_{b1}(t) \\ x'_{b2}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_{11i} & N_{12i} \\ N_{21i} & N_{22i} \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} u_i(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix}$$

The reduced order bilinear system using balanced truncation:

$$x'_{br}(t) = A_{11} x_{br}(t) + \sum_{i=1}^{m} N_{11i} x_{br}(t) u_i(t) + B_1 u(t)$$

$$y_r(t) = C_1 x_{br}(t)$$

Singular Perturbation Approach

$$\begin{bmatrix} x'_{b1}(t) \\ \mu x'_{b2}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_{11i} & N_{12i} \\ N_{21i} & N_{22i} \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} u_i(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \end{bmatrix}$$

$$x_{b2} = -\left(A_{22} + \sum_{i=1}^{m} N_{22i} u_i \right)^{-1} \left(A_{21} + \sum_{i=1}^{m} N_{21i} u_i \right) x_{b1}$$

The reduced-order bilinear system is as follows:

$$x'_{b1}(t) = \left(A_{11} + \sum_{i=1}^{m} N_{11i} u_i(t) - \left(A_{12} + \sum_{i=1}^{m} N_{12i} u_i(t) \right) \left(A_{22} + \sum_{i=1}^{m} N_{22i} u_i(t) \right)^{-1} \left(A_{21} + \sum_{i=1}^{m} N_{21i} u_i(t) \right) \right) x_{b1}(t)$$

$$y(t) = \left(C_1 - C_2 \left(A_{22} + \sum_{i=1}^{m} N_{22i} u_i \right)^{-1} \left(A_{21} + \sum_{i=1}^{m} N_{21i} u_i \right) \right) x_{b1}(t)$$

Stabilization by linear state feedback control

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

By using linear state feedback control $u(t) = Kx(t)$

Find K which minimize

$$J = x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt$$

Robust Control

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + \sum_{k=1}^{m}(N_k + \Delta N_k)x(t)u_k(t) + (B + \Delta B)u(t) + Cw(t) \\ y(t) = Cx(t) \end{cases}$$

By using linear state feedback control $u(t) = Kx(t)$

Find K which minimize

$$\|G\|_\infty$$

Upper Bound of Error Transfer function

Consider a nth-order bilinear system

$$G:\begin{cases}\dot{x}(t)=Ax(t)+\sum_{k=1}^{m}N_k x(t)u_k(t)+Bu(t)\\ y(t)=Cx(t)\end{cases}$$

$$G_r:\begin{cases}\dot{x}_r(t)=A_r x_r(t)+\sum_{k=1}^{m}N_{rk} x_r(t)u_k(t)+B_r u(t)\\ y_r(t)=C_r x_r(t)\end{cases}$$

$r<n$

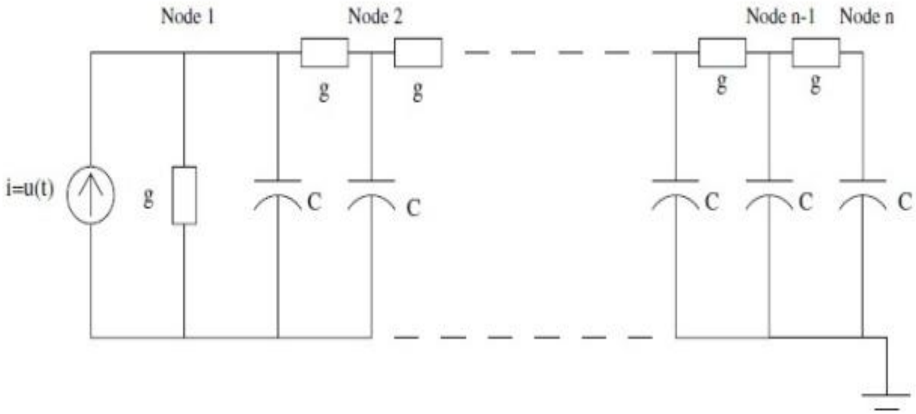
$$\left\|G-G_r\right\|_{\infty}<\sqrt{\lambda_{\max}(P)}\sqrt{\lambda_{\max}(Q)}$$

Genetic Algorithm

1. Initialize random population.
2. Create valid function using grammar.
3. Evaluate fitness value of the chromosome.
4. If the improvement of the fitness tends to zero, stop the procedures. Otherwise proceed to the next step.
5. Generate new population using the genetic operations, go to step2.

Simulation Results

RC Circuit with nonlinear resistor



$$\dot{x}(t)=Ax(t)+N_1x(t)u_1(t)+N_2x(t)u_2(t)+Bu(t)$$
$$y(t)=C^T x(t)$$

$$A=\begin{bmatrix} -5 & 2 & & & \\ 2 & -5 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & -5 & 2 \\ & & & 2 & -5 \end{bmatrix}_{200\times 200}, N_1=\begin{bmatrix} 0 & -3 & & & \\ 3 & 0 & -3 & & \\ & \ddots & \ddots & \ddots & \\ & & 3 & 0 & -3 \\ & & & 3 & 0 \end{bmatrix}_{200\times 200}$$

$$N_2=-N_1+I \qquad B\in R^{200\times 2}, \quad C\in R^{200\times 3}, \quad u(t)=\begin{bmatrix} e^{-t} & e^{-2t} \end{bmatrix}^T$$

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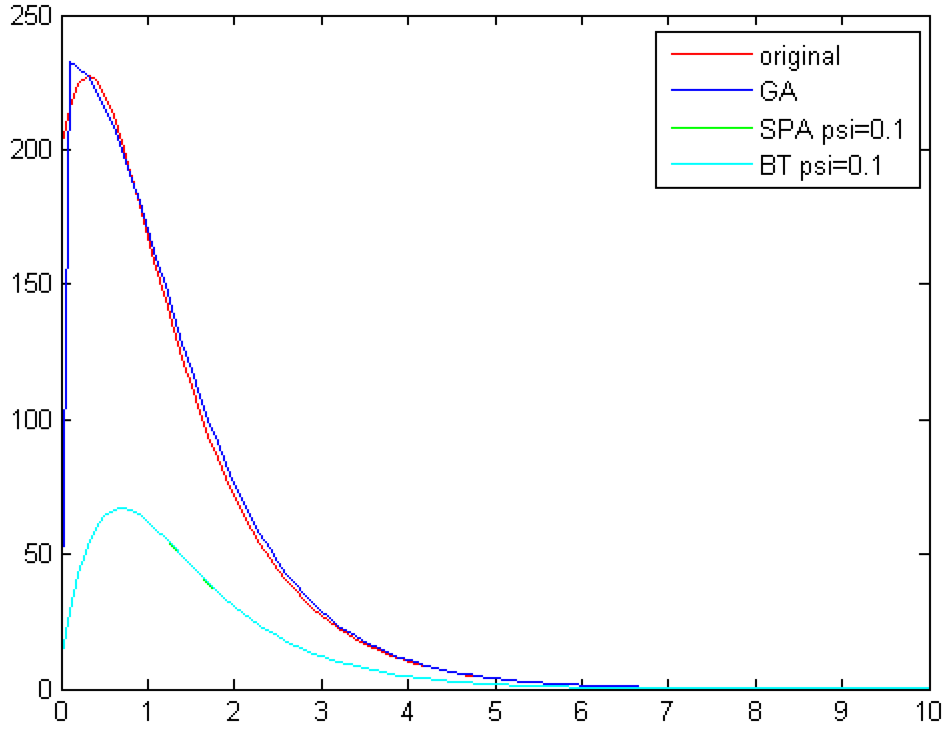
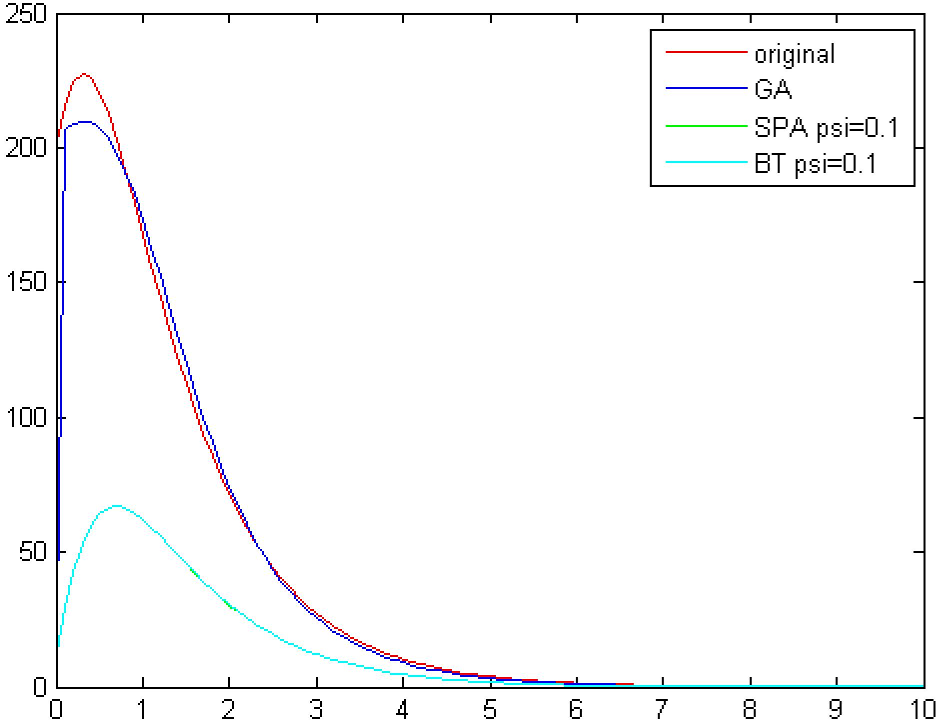
Comparison of the Singular Perturbation (SP) ad Balanced Truncation (BT) and GA

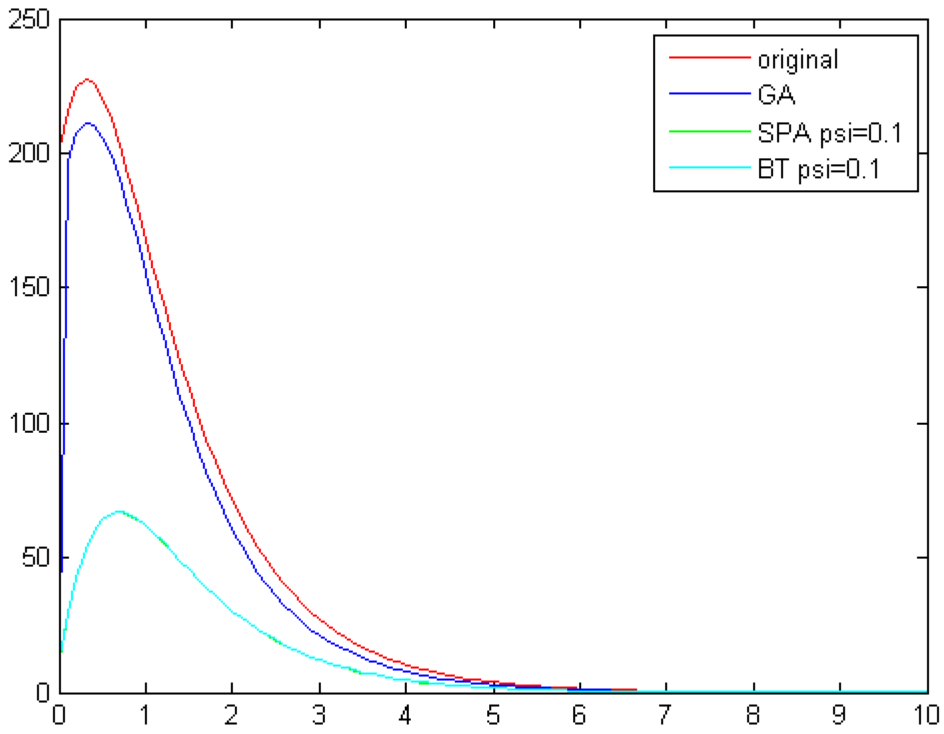


1-st order



250
200
150
100
50
0
0
1
2
3
4
5
6
7
8
9
10
original
GA
SPA psi=0.1
BT psi=0.1

2-nd order



original
GA
SPA psi=0.1
BT psi=0.1
250
200
150
100
50
0
0
1
2
3
4
5
6
7
8
9
10

3-rd order

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Thanks........

Publications

1. Roberd Saragih, Model reduction of bilinear system using genetic algorithm, International Journal of Control and Automation, Vol. 7, No. 5, 2014, pp. 191-200.
2. Roberd Saragih, Computation development for designing bilinear control system, The 10th East Asia SIAM Conference, 23-25 June 2014, Thailand.